Geodesic X-ray transform and streaking artifacts on simple surfaces or on spaces of constant curvature

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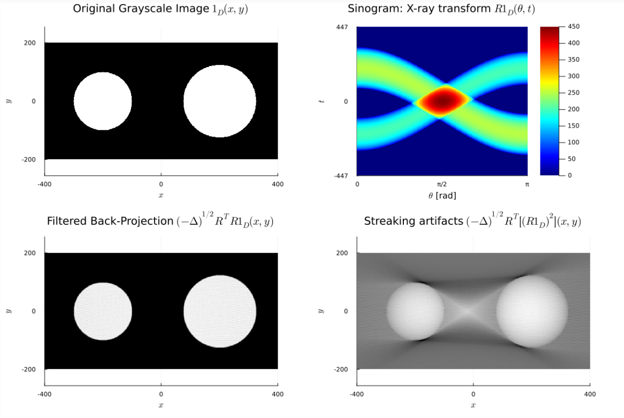
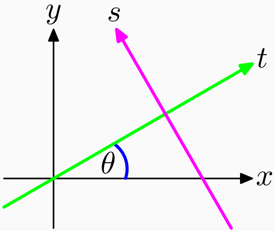
X-ray transform on the plane

- All the planar lines are parametrized by $(\theta, t) \in [0, \pi] \times \mathbb{R}$:
$$\ell = \{(-s\sin\theta + t\cos\theta, s\cos\theta + t\sin\theta) : s \in \mathbb{R}\}.$$
The X-ray transform of $f(x, y)$ on $\mathbb{R}^2$ is defined by
$$\mathcal{R}f(\theta, t) := \int_\ell f = \int_{-\infty}^{\infty} f(-s\sin\theta + t\cos\theta, s\cos\theta + t\sin\theta)ds.$$
This is considered to be the measurements of CT scanners for normal tissue. The FBP formula $f = (-\partial_x^2 - \partial_y^2)^{1/2} \circ \mathcal{R}^T \circ \mathcal{R}f$ is well-known.
- We consider a model of human body f containing a metal region D such as dental implants, stents in blood vessels, and etc. We observe that the metal streaking artifacts caused by beam hardening effect in the energy level of X-ray. The main term is the filtered back-projection of nonlinear term
$$(-\partial_x^2 - \partial_y^2)^{1/2} \circ \mathcal{R}^T\left[(\mathcal{R}1_D)^2\right],$$
This is a conormal distribution whose singular support is the streaking artifact.

Figures: metal streaking artifacts





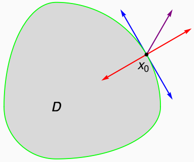
The main part of artifacts: $(-\partial_x^2 - \partial_y^2)^{1/2}\mathcal{R}^T\left[(\mathcal{R}1_D)^2\right]$.

Conormal distributions

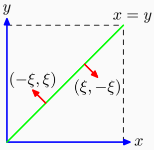
Definition 1 (Conormal distributions)

Let X be an N-dim manifold, and let Y be a closed submanifold of X. We say that $u \in \mathscr{D}'(X)$ is conormal with respect to Y of degree m if $L_1 \cdots L_\mu u \in {}^\infty H^{\mathrm{loc}}_{(-m-N/4)}(X)$ for all $\mu = 0, 1, 2, \ldots$ and all vector fields $L_1, \ldots, L_\mu$ tangential to Y. Denote by $I^m(N^*(Y))$ the set of all distributions on X conormal wrt Y of degree m. Note that $\mathrm{WF}(u) \subset N^*(Y) \setminus 0$.

- The characteristic function of a domain: $1_D \in I^{-1/2-n/4}(N^*(\partial D))$ for $D \subset \mathbb{R}^n$, which is a domain with smooth boundary.



- The Schwartz kernel of a PsDO: $\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x,\xi)\, d\xi \in I^m(N^*(\Delta))$, $\Delta = \{(x,x)\}$ for $a(x,\xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$.



Geodesic X-ray transform 1

Suppose that (M, g) is a compact nontrapping simple Riemannian manifold with strictly convex smooth boundary. A map $\pi : S(M) \to M$ is the natural projection. Denote by $\partial_- S(M)$ the set of all unit incoming tangent vectors on the boundary ∂M:

$$\partial_- S(M) = \{w \in S(M) : \pi(w) \in \partial M, \langle \nu, w \rangle < 0\},$$

where $\nu(x)$ is the unit outer normal vector at $x \in \partial M$. Note that the nontrapping condition ensures that $\partial_- S(M)$ is identified with the manifold of all the normal geodesics on (M, g):

$$\partial_- S(M) \simeq \mathcal{G} := \{\gamma_v : \ \nabla_{\dot\gamma_v(t)} \dot\gamma_v(t) = 0, \ \dot\gamma_v(0) = v \in S(M)\}.$$

The geodesic X-ray transform of a function (more precisely a half-density) f on M is defined by

$$\mathcal{X} f(w) := \int_0^{\tau(w)} f\big(\gamma_w(s)\big)\, ds, \quad w \in \partial_- S(M),$$

where $\tau(w)$ is the exit time of γ_w.

Geodesic X-ray transform 2

Set $n = \dim(M)$. Then $\dim\big(S(M)\big) = 2n - 1$ and $\dim\big(\partial_- S(M)\big) = 2n - 2$.

Let $F : S(M) \to \partial_- S(M)$ be the submersion defined by $F\big(\dot\gamma_w(t)\big) = w$ for $w \in \partial_- S(M)$ and $t \in [0, \tau(w)]$. Then we have $\mathcal{X} = F_* \circ \pi^*$ and $\mathcal{X}^T = \pi_* \circ F^*$. See Holman-Uhlmann (2018).

Proposition 2

$\mathcal{X}$ is an elliptic Fourier integral operator, and its Schwartz kernel belongs to

$$I^{-n/4}\big(\partial_- S(M) \times M^{int}, C'_{\mathcal{X}}; \Omega^{1/2}_{\partial_- S(M) \times M^{int}}\big),$$

where $C_{\mathcal{X}}$ is the canonical relation of $\mathcal{X}$: we say that $(\xi, \eta) \in C_{\mathcal{X}}$ if $\exists v \in S(M^{int})$ such that

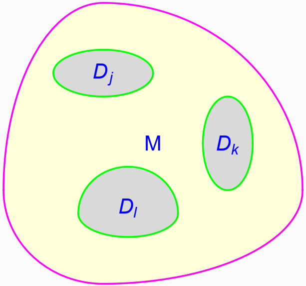
$$\xi \in T^*_{F(v)}\big(\partial_- S(M)\big) \setminus \{0\}, \quad \eta \in T^*_{\pi(v)}(M^{int}) \setminus \{0\}, \quad DF|_v^T \xi = D\pi|_v^T \eta,$$

Assumption 1

- Assume that $\dim(M) = 2$ or (M, g) is a space of constant curvature.

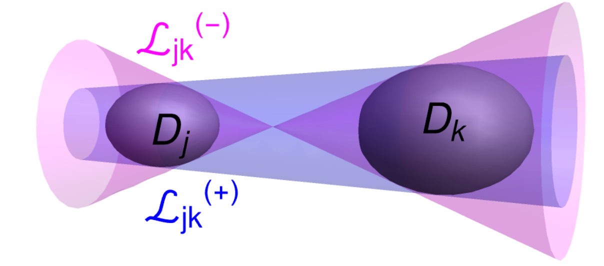
  This ensures that all the Jacobi fields are of the form scalar function $\times$ parallel transport.

- Suppose that the metal region $D \subset M^{\text{int}}$ is a disjoint union of D_j $(j = 1 \ldots, J)$ which are simply connected, strictly convex and bounded with smooth boundaries ∂D_j.



A hypersurface $\mathscr{L}$ surrounding the metal region D

- For any j and $x \in \partial D_j$, denote by $\nu_j(x)$ the unit outer normal vector at x. Consider the tangent hyperplane $\exp_x \nu_j(x)^\perp \cap M^{\text{int}}$ at $x \in \partial D_j$.
- There are some common tangent hyperplanes of ∂D_j and ∂D_k for $j \neq k$. In this case there is common tangent geodesics in such hyperplanes. The union of all these geodesics forms a conical or cylindrical hypersurface denoted by $\mathscr{L}_{jk}^{(\pm)}$. Set $\mathscr{L} := \bigcup \left(\mathscr{L}_{jk}^{(+)} \cup \mathscr{L}_{jk}^{(-)} \right)$.



Assumption 2 (The simple model of beam hardening effect)

Let $E \geqq 0$ be a parameter describing the energy level of the X-ray beam, and let E_0 be the fixed standard level for the normal tissue. The measurement P is of the form:

$$P = -\log \left\{ \int_0^\infty \rho(E) \exp(-\mathcal{X} f_E) dE, \right\},$$

where $\rho(E)$ is a probability density function on $[0, \infty)$ and is called the spectral function. Let f_{CT} be the FBP of P. We employ the simple model of the form

$$f_E(x) = f_{E_0}(x) + \alpha(E - E_0) 1_D(x), \quad \rho(E) = \frac{1}{2\varepsilon} 1_{[E_0 - \varepsilon, E_0 + \varepsilon]}(E)$$

with small parameters $\alpha > 0$ and $\varepsilon > 0$.

Main Theorem

Then the nonlinear effect f_{MA} in the CT image becomes

$$f_{\mathrm{MA}} := f_{\mathrm{CT}} - f_{E_0} = \sum_{k=1}^{\infty} (\alpha\varepsilon)^{2k} A_k Q \mathcal{X}^T [(\mathcal{X} 1_D)^{2k}] \mod C^{\infty}(M^{\mathrm{int}}), \quad \{A_k\} \subset \mathbb{R},$$

where Q is a parametrix of $\mathcal{X}^T \circ \mathcal{X}$: $Q\mathcal{X}^T\mathcal{X} = Id$ modulo smoothing operators locally.

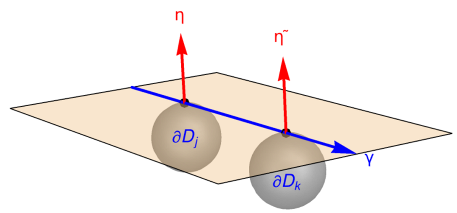
Our main result is as follows:

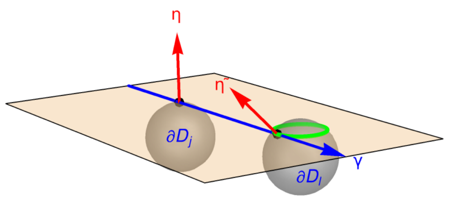
Theorem 3

$f_{MA} \in I^{-3n/4-1/2}(N^*(\mathscr{L}))$ away from ∂D, and $\sigma_{prin}(Q\mathcal{X}^T[(\mathcal{X}1_D)^2]) \neq 0$.

- Park-Choi-Seo (2017) proved that $\mathrm{WF}(f_{\mathrm{MA}}) \subset N^*(\mathscr{L})$ for $M = \mathbb{R}^2$.
- Palacios-Uhlmann-Wang (2018) proved Theorem 3 for $M = \mathbb{R}^2$.
- C (2022) proved Theorem 3 for the d-plane transform on $\mathbb{R}^n$.
  We could NOT understand the meaning in many parts of this paper.

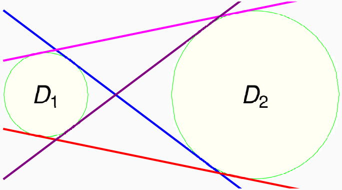
- If ∂D_j and ∂D_k have a common tangent hyperplane, then the conormal singularities propagate along the common tangent geodesic. See the left figure.
- Suppose $n \geqq 3$. If ∂D_j and ∂D_k have a common tangent geodesic, but the conormal directions at the tangent points are different, then the conormal singularities do not propagate along the common tangent geodesic. See the right figure.

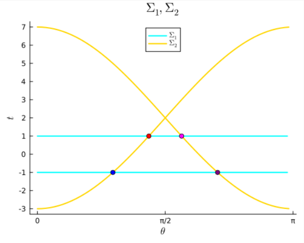




Outline of the proof of Theorem 3

- $1_{D_j} \in I^{-1/2-n/4}(N^*(\partial D_j) \setminus 0)$.
- $\mathcal{X}1_{D_j} \in I^{-(n+1)/2}(N^*(\Sigma_j) \setminus 0)$ with some hypersurface Σ_j in $\partial_- S(M)$.
- For $j \neq k$, Σ_j is transversal to Σ_k.





- Set $\Sigma_{jk} := \Sigma_j \cap \Sigma_k$ for short. For $j \neq k$,

$$\mathcal{X}1_{D_j} \cdot \mathcal{X}1_{D_k} \in \begin{cases} I^{-(n+1)/2-1}(N^*(\Sigma_{jk}) \setminus 0) & \text{at } \Sigma_{jk}, \\ I^{-(n+1)/2}(N^*(\Sigma_j) \setminus 0) + I^{-(n+1)/2}(N^*(\Sigma_k) \setminus 0) & \text{away from } \Sigma_{jk}. \end{cases}$$

Key: $C_{\mathcal{X}^T} \circ \Sigma_{jk} \setminus 0 = N^*(\mathcal{L}_{jk}) \setminus 0$

- Fix arbitrary geodesic $\gamma_w \simeq w \in \Sigma_{jk}$.
- If $\xi, \tilde{\xi} \in T^*_w(\partial_- S(M))$, $\quad w = F(v) = F(\tilde{v})$, $\quad \pi(v) \in \partial D_j$, $\quad \pi(\tilde{v}) \in \partial D_k$,

$$DF|_v^T \xi = D\pi|_v^T \eta, \quad \eta \in N^*_v(\partial D_j) \setminus \{0\}, \quad DF|_{\tilde{v}}^T \tilde{\xi} = D\pi|_{\tilde{v}}^T \tilde{\eta}, \quad \tilde{\eta} \in N^*_{\tilde{v}}(\partial D_k) \setminus \{0\},$$

  then ξ and $\tilde{\xi}$ are linearly independent, and the nonlinear effect on the geodesic γ_w creates two-dimensional singularity span$\langle \xi, \tilde{\xi} \rangle$ in $T^*_w(\partial_- S(M))$ due to the simplicity condition.
- WLOG WMA η and $\tilde{\eta}$ are unit covectors.
- WLOG WMA η is the parallel transport of $\tilde{\eta}$ if $\eta \parallel \tilde{\eta}$.
- We shall show that if $\tilde{\eta}$ is the parallel transport of η, then

$$C^T_{\mathcal{X}} \circ \operatorname{span}\langle \xi, \tilde{\xi} \rangle = \bigcup_{a \in \mathbb{R}} (\text{the parallel transport of } \eta \text{ along } \gamma_w) = \bigcup_{t \in [0, \tau(w)]} N^*_{\gamma_w(t)}(\mathcal{L}_{jk}),$$

  otherwise, $C^T_{\mathcal{X}} \circ \operatorname{span}\langle \xi, \tilde{\xi} \rangle = N^*_{\pi(v)}(\partial D_j) \bigcup N^*_{\pi(\tilde{v})}(\partial D_k)$.

When $\tilde{\eta}$ is the parallel transport of η

- Set $\gamma_w(t_0) = \pi(\eta) \in \partial D_j$ and $\gamma_w(\tilde{t}_0) = \pi(\tilde{\eta}) \in \partial D_k$, and suppose $\tilde{\eta} = P(\tilde{t}_0, t_0; \gamma_w)^T \eta$, where $P(t_0, \tilde{t}_0; \gamma_w)$ is the parallel transport of $T_{\gamma_w(\tilde{t}_0)}(M)$ onto $T_{\gamma_w(t_0)}(M)$ along γ_w. Set $\eta(s) := P(s, t_0; \gamma_w)^T \eta \in T^*_{\gamma_w(s)}(M^{\mathrm{int}})$ for $s \in (0, \tau(w))$. Then $\eta(\tilde{t}_0) = \tilde{\eta}$.
- Let $k(x)$ be a sectional curvature at $x \in M$, which is a constant when $n \geqq 3$.
- Let $a(t; s), b(t; s) \in C^\infty(0, \tau(w))$ be solutions to

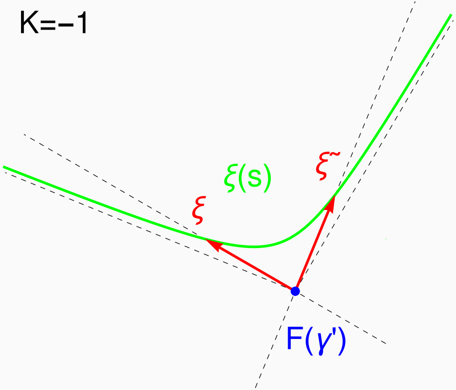
$$
\begin{aligned}
a_{tt}(t; s) + k\big(\gamma_w(t)\big)a(t; s) &= 0, \quad a(s; s) = 1, \quad a_t(s; s) = 0,\\
b_{tt}(t; s) + k\big(\gamma_w(t)\big)b(t; s) &= 0, \quad b(s; s) = 0, \quad b_t(s; s) = 1.
\end{aligned}
$$

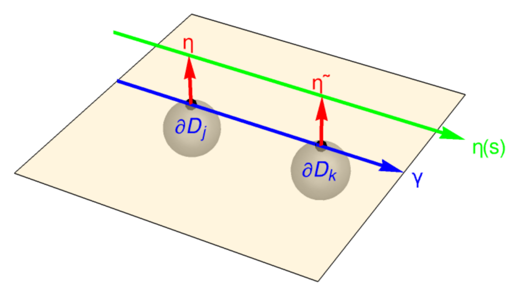
- $\Delta(t_0, \tilde{t}_0; s) := \det \begin{bmatrix} a(t_0; s) & a(\tilde{t}_0; s) \\ b(t_0; s) & b(\tilde{t}_0; s) \end{bmatrix}$ never vanish due to the simplicity. If we set

$$
\xi(s) := \frac{b(\tilde{t}_0; s)}{\Delta(t_0, \tilde{t}_0; s)}\xi - \frac{b(t_0; s)}{\Delta(t_0, \tilde{t}_0; s)}\tilde{\xi} \in \mathrm{span}\langle \xi, \tilde{\xi} \rangle, \quad s \in (0, \tau(w)),
$$

then we have $DF|^T_{\dot{\gamma}_w(s)}\xi(s) = D\pi|^T_{\dot{\gamma}_w(s)}\eta(s)$ in $T^*_{\dot{\gamma}_w(s)}\big(S(M^{\mathrm{int}})\big)$ for $s \in (0, \tau(w))$.

$\xi(s)$ in span$\langle \xi, \tilde{\xi} \rangle \subset T^*_{F(\gamma)}(\partial_- S(M))$ and $\eta(s)$ in $T^*(S(M^{\mathrm{int}}))$ for $K = -1$





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