

# Normal operators of double fibration transforms

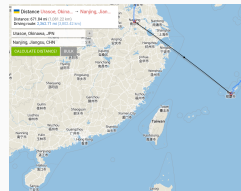
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## Double fibration transforms



# Double fibration

Following Mazzucchelli-Salo-Tzou [3], we introduce double fibration transforms.

- Let  $\mathcal{G}$  and  $X$  be oriented smooth manifolds without boundaries.  $N := \dim(\mathcal{G})$  and  $n := \dim(X)$ . Denote by  $d\mathcal{G}$  and  $dX$  the orientation forms of  $\mathcal{G}$  and  $X$  respectively.
- Let  $Z$  be an oriented embedded submanifold of  $\mathcal{G} \times X$ , and let  $dZ$  be the orientation form.
- Assume that  $N + n > \dim(Z) > N \geq n \geq 2$ , and set  $n' := \dim(Z) - N$  and  $n'' := n - n'$ . Then  $\dim(Z) = N + n'$ ,  $n = n' + n''$  and  $n', n'' = 1, \dots, n - 1$ .

$$\begin{array}{ccc} & Z & \\ \pi_{\mathcal{G}} \swarrow & & \searrow \pi_X \\ \mathcal{G} & & X \end{array} \qquad \begin{array}{ccc} & (N^*Z \setminus 0)' & \\ \pi_L \swarrow & & \searrow \pi_R \\ T^*\mathcal{G} \setminus 0 & & T^*X \setminus 0 \end{array}$$

- We assume that  $Z$  is a **double fibration**, that is, the natural projections  $\pi_{\mathcal{G}} : Z \rightarrow \mathcal{G}$  and  $\pi_X : Z \rightarrow X$  are submersions respectively.

## Orientation forms on $G_z := \pi_X \circ \pi_{\mathcal{G}}^{-1}(z)$ and $H_x := \pi_{\mathcal{G}} \circ \pi_X^{-1}(x)$

- $G_z := \pi_X \circ \pi_{\mathcal{G}}^{-1}(z)$  becomes an  $n'$ -dim submanifold of  $X$  for any  $z \in \mathcal{G}$ , and  $H_x := \pi_{\mathcal{G}} \circ \pi_X^{-1}(x)$  forms an  $(N - n'')$ -dim submanifold of  $\mathcal{G}$  for any  $x \in X$ .
- Fix arbitrary  $(z, x) \in Z$ , and let  $\{v_1, \dots, v_{n'}\}$  and  $\{w_1, \dots, w_N\}$  be bases of  $T_x X$  and  $T_z \mathcal{G}$  respectively such that

$$T_{(z,x)}Z = \text{span}\langle v_1, \dots, v_{n'}, w_1, \dots, w_N \rangle = \text{span}\langle v_1, \dots, v_{n'}, w_1, \dots, w_{N-n''} \rangle.$$

The induced orientation forms  $dG_z$  on  $G_z$  and  $dH_x$  on  $H_x$  are given by

$$\begin{aligned} dG_z(d\pi_X(v_1), \dots, d\pi_X(v_{n'})) &:= dZ_{\pi_{\mathcal{G}}^{-1}(z)}(v_1, \dots, v_{n'}) \\ &= \frac{dZ(v_1, \dots, v_{n'}, w_1, \dots, w_N)}{d\mathcal{G}(d\pi_{\mathcal{G}}(w_1), \dots, d\pi_{\mathcal{G}}(w_N))}, \\ dH_x(d\pi_{\mathcal{G}}(w_1), \dots, d\pi_X(w_{N-n''})) &:= dZ_{\pi_X^{-1}(x)}(w_1, \dots, w_{N-n''}) \\ &= \frac{dZ(v_1, \dots, v_{n'}, w_1, \dots, w_{N-n''})}{dX(d\pi_X(v_1), \dots, d\pi_X(v_{n'}))}. \end{aligned}$$

# Double fibration transform

Suppose that a weight function  $\kappa(z, x) \in C^\infty(\mathcal{G} \times X)$  is nowhere vanishing. A double fibration transform  $\mathcal{R}$  associated with the double fibration  $Z$  is defined by

$$\mathcal{R}f(z) := \left( \int_{G_z} \kappa(z, x) \frac{f}{|dX|^{1/2}}(x) dG_z(x) \right) |d\mathcal{G}(z)|^{1/2}$$

for  $f \in \mathcal{D}(X, \Omega_X^{1/2})$ . The adjoint  $\mathcal{R}^*$  is given by

$$\mathcal{R}^*u(x) = \left( \int_{H_x} \overline{\kappa(z, x)} \frac{u}{|d\mathcal{G}|^{1/2}}(z) dH_x(z) \right) |dX(x)|^{1/2}$$

for  $u \in \mathcal{D}(\mathcal{G}, \Omega_{\mathcal{G}}^{1/2})$ . Then we deduce that

$$\mathcal{R} : \mathcal{D}(X, \Omega_X^{1/2}) \rightarrow \mathcal{E}(\mathcal{G}, \Omega_{\mathcal{G}}^{1/2}), \quad \mathcal{R}^* : \mathcal{D}(\mathcal{G}, \Omega_{\mathcal{G}}^{1/2}) \rightarrow \mathcal{E}(X, \Omega_X^{1/2}),$$

are continuous linear mappings, and so are

$$\mathcal{R} : \mathcal{E}'(X, \Omega_X^{1/2}) \rightarrow \mathcal{D}'(\mathcal{G}, \Omega_{\mathcal{G}}^{1/2}), \quad \mathcal{R}^* : \mathcal{E}'(\mathcal{G}, \Omega_{\mathcal{G}}^{1/2}) \rightarrow \mathcal{D}'(X, \Omega_X^{1/2}).$$

More precisely  $\mathcal{R}$  and  $\mathcal{R}^*$  are elliptic Fourier integral operators.

# Mapping properties of double fibration transforms

## Theorem 1 ([3, Theorem 2.2] & [Hörmander IV, Theorem 25.2.2])

*Suppose that  $Z$  is a double fibration with  $\dim(Z) = N + n'$ . Then  $\mathcal{R}$  and  $\mathcal{R}^*$  are elliptic Fourier integral operators of order  $-(N + 2n' - n)/4$  with canonical relations  $(N^*Z \setminus 0)'$  and  $((N^*Z \setminus 0)^T)'$  respectively. More precisely*

$$\begin{aligned}\mathcal{R} &\in \mathcal{I}^{-(N+2n'-n)/4}(\mathcal{G} \times X, N^*Z \setminus 0; \Omega_{\mathcal{G} \times X}^{1/2}), \\ \mathcal{R}^* &\in \mathcal{I}^{-(N+2n'-n)/4}(X \times \mathcal{G}, (N^*Z \setminus 0)^T; \Omega_{X \times \mathcal{G}}^{1/2}),\end{aligned}$$

where

$$\begin{aligned}N^*Z \setminus 0 &= \{ (z, A(z, x)\eta, x, \eta) : (z, x) \in Z, \eta \in N_x^*G_z \setminus \{0\} \} \\ &= \{ (z, \zeta, x, B(z, x)\zeta) : (z, x) \in Z, \zeta \in N_z^*H_x \setminus \{0\} \},\end{aligned}$$

$A(z, x) \in \text{Hom}(N_x^*G_z, T_z^*\mathcal{G})$  and  $B(z, x) \in \text{Hom}(N_z^*H_x, T_x^*X)$  smoothly depend on  $(z, x) \in Z$  respectively.

# Examples of double fibration transforms

- $d$ -plane transform on  $\mathbb{R}^n$ :  $\mathcal{G} :=$  the set of all  $d$ -dimensional planes in  $\mathbb{R}^n$ ,  $X := \mathbb{R}^n$ .
- Geodesic X-ray transform: Let  $(M, g)$  be a compact and nontrapping Riemannian manifold with a strictly convex boundary. Denote by  $\nu(x)$  the unit outer normal vector at  $x \in \partial M$ . Then

$$\mathcal{G} := \partial_- SM = \{(x, u) \in SM : x \in \partial M, \langle u, \nu(x) \rangle < 0\}, \quad X := M^{\text{int}}.$$

Finsler manifolds  $(M, F)$ ?: Probably,  $F(x, y) = F(x, -y)$  is required.

- Null bicharacteristics: Let  $P$  be a real-principal-type pseudodifferential operator on a manifold  $X$  with the principal symbol  $p_m(x, \xi)$ .

$$\mathcal{G} := \pi_X(\text{all Hamilton flows for } p_m(x, \xi) = 0).$$

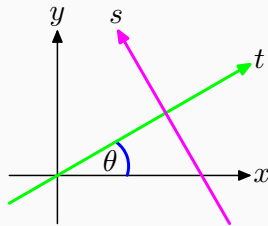
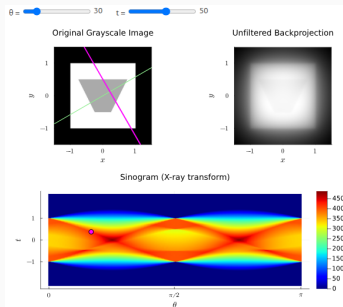
Light ray transform is a typical example.

# X-ray transform on $\mathbb{R}^2$

Let  $n = 2$ . The X-ray transform of a function  $f$  of  $(x, y) \in \mathbb{R}^2$  is defined by

$$R_1 f(\theta, t) := \int_{-\infty}^{\infty} f(t \cos \theta - s \sin \theta, t \sin \theta + s \cos \theta) ds, \quad (\theta, t) \in [0, \pi] \times \mathbb{R}.$$

$R_1^* R_1$  is the normal operator (unfiltered backprojection) of  $R_1$  and  $R_1^* R_1 = (-\partial_x^2 - \partial_y^2)^{-1/2}$ .  
 $(-\partial_x^2 - \partial_y^2)^{1/2} R_1^* R_1 f = f$  is known as the filtered backprojection.



## Z-conjugate points

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For local coordinates  $(z, x) = (z', z'', x', x'') \in \mathbb{R}^{N-n''} \times \mathbb{R}^{n''} \times \mathbb{R}^{n'} \times \mathbb{R}^{n''}$ , There exist  $\mathbb{R}^{n''}$ -valued functions  $\phi(z, x')$  and  $b(x, z')$  such that we have locally

$$Z = \{x'' = \phi(z, x')\} = \{z'' = b(x, z')\}.$$

**Lemma 2 ([3, Lemmas 2.4, 2.5 and 2.6])**

$$N_{(z,x)}^* Z = \left\{ \left( -\phi_z(z, x')^T \eta'', (-\phi_{x'}(z, x')^T \eta'', \eta'') \right) : \eta'' \in \mathbb{R}^{n''} \right\},$$

$$A(z, x) \begin{bmatrix} -\phi_{x'}(z, x')^T \\ I_{n''} \end{bmatrix} \eta'' = -\phi_z(z, x')^T \eta'', \quad \eta'' \in \mathbb{R}^{n''}.$$

*Similar results hold for  $b(x, z')$  and  $B(z, x)$ .*

## Variation fields and conjugate points

Fix arbitrary  $(z, w) \in T\mathcal{G}$ , we introduce the variation field  $J_w : G_z \rightarrow (N_x^* G_z)^*$  is defined by

$$J_w(x) := A(z, x)^* w \simeq -\phi_z(z, x') w \in (N_x^* G_z)^* \simeq N_x G_z = T_x X / T_x G_z$$

for  $x \in G_z$ . Note that

$$A(z, x)^* \in \text{Hom}(T_z \mathcal{G}, (N_x^* G_z)^*) \simeq \text{Hom}(T_z \mathcal{G}, N_x G_z), \quad (z, x) \in Z.$$

For  $z \in \mathcal{G}$  and  $x, y \in G_z$ , set

$$V_z(x, y) := \{J_w(x) : w \in T_z \mathcal{G}, J_w(y) = 0\}.$$

Note that  $\dim(V_z(x, y)) \leq n''$  holds since  $\text{rank}(A(z, x)^*) = n''$ .

Riemannian manifold case: If  $x = \exp_y(tu)$ , then  $J_w(x) \simeq Y(t) := tD \exp_y(tu)w$ .

## Definition 3

Suppose that  $Z$  is a double fibration and  $N \geq 2n''$ . Let  $k = 1, \dots, n''$ .

- **Z-conjugate triplet of degree  $k$ :** Let  $z \in \mathcal{G}$  and let  $x, y \in G_z$  with  $x \neq y$ . We say that  $(z; x, y)$  is a  $Z$ -conjugate triplet of degree  $k$  if  $\dim(V_z(x, y)) = n'' - k$ .
- **Regular Z-conjugate triplet of degree  $k$ :** We say that a  $Z$ -conjugate triplet  $(z; x, y)$  of degree  $k$  is regular if there exist a nbd  $U_x$  of  $x$  in  $X$ , a nbd  $U_y$  of  $y$  in  $X$ , and a nbd  $W_z$  of  $z$  in  $\mathcal{G}$  such that any  $Z$ -conjugate triplet  $(z'; x', y') \in W_z \times U_x \times U_y$  is also of degree  $k$ . The set of all the regular  $Z$ -conjugate triplets of degree  $k$  is denoted by  $C_{R,k}$ .
- The set of all the  $Z$ -conjugate triplets which are not regular is denoted by  $C_S$ .

## How to describe $Z$ -conjugacy 1/2

Let  $(z_0; x_0, y_0)$  be a  $Z$ -conjugate triplet of degree  $k = 1, \dots, n''$ .  $Z$  is locally expressed by  $\phi$  and  $\psi$  as  $\{x'' = \phi(z, x')\}$  near  $(z_0, x_0)$  and  $\{y'' = \psi(z, y')\}$  near  $(z_0, y_0)$  respectively, and

$$V_{z_0}(x_0, y_0) = \{\phi_z(z_0, x'_0)w : w \in \text{Ker}(\psi_z(z_0, y'_0))\}.$$

Set  $\phi = [\phi^{(1)}, \dots, \phi^{(n'')}]^T$  and  $\psi = [\psi^{(1)}, \dots, \psi^{(n'')}]^T$ . Then  $\{\phi_z^{(1)}, \dots, \phi_z^{(n'')}\}$  and  $\{\psi_z^{(1)}, \dots, \psi_z^{(n'')}\}$  are linearly independent near  $(z_0, x_0)$  and  $(z_0, y_0)$  respectively. Note that

$$\text{Ker}(\psi_z(z_0, y'_0)) = \text{span}\langle \psi_z^{(1)}(z_0, y'_0)^T, \dots, \psi_z^{(n'')}(z_0, y'_0)^T \rangle^\perp \quad \text{in } \mathbb{R}^N.$$

We deduce that  $\dim(V_{z_0}(x_0, y_0)) = n'' - k$  is equivalent to

$$\dim\left(\text{span}\langle \phi_z^{(1)}(z_0, x'_0), \dots, \phi_z^{(n'')}(z_0, x'_0) \rangle \cap \text{span}\langle \psi_z^{(1)}(z_0, y'_0), \dots, \psi_z^{(n'')}(z_0, y'_0) \rangle\right) = k. \quad (1)$$

We express (1) by  $k$  equations.

## How to describe $Z$ -conjugacy 2/2

Denote by  $\{\tilde{\psi}_z^{(1)}, \dots, \tilde{\psi}_z^{(n'')}\}$  the Schmidt orthonormalization of  $\{\psi_z^{(1)}, \dots, \psi_z^{(n'')}\}$ .

(1) is equivalent to the following: there exist  $\lambda_1, \dots, \lambda_k \in \mathbb{R}^{n''}$ ,  $\lambda_l = [\lambda_{l1}, \dots, \lambda_{ln''}]$  ( $l = 1, \dots, k$ ) such that  $\lambda_1, \dots, \lambda_k$  are linearly independent and if we set

$$\begin{aligned}\phi_z^{\lambda_l}(z, x') &:= \lambda_l \phi_z(z, x') = \sum_{m=1}^{n''} \lambda_{lm} \phi_z^{(m)}(z, x'), \\ H^{\lambda_l}(z, x', y') &:= \phi_z^{\lambda_l}(z, x') \phi_z^{\lambda_l}(z, x')^T - \sum_{m=1}^{n''} |\phi_z^{\lambda_l}(z, x') \tilde{\psi}_z^{(m)}(z, y')^T|^2 \\ &= \lambda_l \phi_z(z, x') (I_N - \tilde{\psi}_z^T(z, y') \tilde{\psi}_z(z, y')) \phi_z(z, x')^T \lambda_l^T, \\ H^\lambda(z, x', y') &:= [H^{\lambda_1}(z, x', y'), \dots, H^{\lambda_k}(z, x', y')]^T,\end{aligned}$$

for  $l = 1, \dots, k$ , then

$$H^\lambda(z_0, x'_0, y'_0) = 0. \quad (2)$$

## An artificial condition (H)

**Condition (H):** Let  $k = 1, \dots, n''$ . Suppose that  $(z_0; x_0, y_0) \in C_{R,k}$ . Suppose that  $H^\lambda(z, x', y')$  is the same as that of the previous paragraph and satisfies (2).

Condition (H) is that  $\text{rank}(D_{z,x',y'} H^\lambda(z_0, x'_0, y'_0)) = 1$  holds for any choice of linearly independent  $\lambda_1, \dots, \lambda_k \in \mathbb{R}^{n''}$ .

### Spirit of Condition (H):

- **rank one:** We refer the case of the geodesic X-ray transform for the dimension. In other words we fit our case to the geodesic X-ray transform.
- **for any choice of  $\lambda_1, \dots, \lambda_k$ :** We avoid the case that there exist two choices  $\lambda_1, \dots, \lambda_k$  and  $\lambda'_1, \dots, \lambda'_k$  such that

$$\{H^\lambda(z, x', y') = 0\} \cap \{H^{\lambda'}(z, x', y') = 0\}$$

is transversal.

# Lemma 4

Suppose that  $Z$  is a double fibration,  $N \geq 2n''$  and some condition (H). For any  $k = 1, \dots, n''$ ,  $C_{R,k}$  is an  $(N + 2n' - 1)$ -dimensional embedded submanifold of  $\mathcal{G} \times X \times X$ .

Note that the connected component of  $C_{R,k}$  containing  $(z_0; x_0, y_0)$  is characterized by

$$F(x'', y'', z; x', y') := \begin{bmatrix} x'' - \phi(z, x') \\ y'' - \psi(z, y') \\ H^\lambda(z, x', y') \end{bmatrix} = 0 \quad \text{near} \quad (z_0; x_0, y_0).$$

We have

$$\text{rank}(DF(x_0'', y_0'', z_0; x_0', y_0')) = \text{rank} \begin{bmatrix} I_{n''} & O & * \\ O & I_{n''} & * \\ O & O & D_{z, x', y'} H^\lambda(z_0, x_0', y_0') \end{bmatrix} = 2n'' + 1.$$

# Structure of normal operators

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## Normal operators without $Z$ -conjugate points

$$\mathcal{R}^* \mathcal{R} f(x) = \left( \iint_{H_x \times G_z} \overline{\kappa(z, x)} \kappa(z, y) \frac{f}{|dX|^{1/2}}(y) dG_z(y) dH_x(z) \right) |dX(x)|^{1/2}, \quad x \in X.$$

Set  $\mathcal{C} := (N^*Z \setminus 0)'$ , which is the canonical relation of  $\mathcal{R}$ .

### Theorem 5

*Suppose that  $Z$  is a double fibration. In addition, we assume  $N \geq 2n''$  and the following:*

- $\pi_X : Z \rightarrow X$  is proper, and  $\pi_X^{-1}(x)$  is connected for any  $x \in X$ .*
- There are no  $Z$ -conjugate triplets, and  $D\pi_L$  is injective at all  $(z, \zeta, x, \eta) \in \mathcal{C}$ .*

*Then  $\mathcal{C}^T \circ \mathcal{C}$  is a clean intersection with excess  $e = N - n$ , and  $\mathcal{R}^* \mathcal{R}$  is an elliptic pseudodifferential operator of order  $-n'$  on  $X$ .*

Proof: Some lemmas in [3] and the assumptions guarantee the Bolker condition. □

# Known results on geodesic X-ray transforms with conjugate points

- Let  $(M, g)$  be a compact Riemannian manifold with strictly convex boundary. Set  $\gamma_w(t) := \exp_{\pi_M(w)}(tw)$  for  $w \in \partial_- SM$ . Consider the geodesic X-ray transform

$$\mathcal{X}f(w) := \left( \int_0^{\tau(w)} \kappa(\gamma_w(t), \dot{\gamma}_w(t)) \frac{f}{|dM|^{1/2}}(\gamma_w(t)) dt \right) |d\partial_- SM(w)|^{1/2}.$$

- Stefanov and Uhlmann (2012) [4]: If  $v_0 = |v_0|\theta_0$  is a fold conjugate vector at  $p_0$ , and  $v_0$  is the only singularity of  $\exp_{p_0}(v)$  on  $\gamma_{\theta_0}$  near  $p_0$ , then the localized normal operator is decomposed as

$$\mathcal{X}^* \chi \mathcal{X} = A + F \quad \text{near } p_0,$$

where  $A$  is a PsDOs of order  $-1$ , and  $F$  is a *FIO* of order  $-n/2$ .

- Holman and Uhlmann (2018) [2]: If  $C_S = \emptyset$ , then

$$\mathcal{X}^* \mathcal{X} = A + \sum_{k=1}^{n-1} \sum_{\alpha=1}^{M_k} F_{k,\alpha},$$

where  $A$  is a PsDOs of order  $-1$ , and  $F$  is a *FIO* of order  $-(n - k + 1)/2$ .

# Normal operators with $Z$ -conjugate points

## Theorem 6

Suppose that  $Z$  is a double fibration,  $C_S = \emptyset$ ,  $N \geq 2n''$ , condition (H) and the following:

- $\pi_X$  is proper, and  $\pi_X^{-1}(x)$  is connected for any  $x \in X$ .
- If  $\pi_L^{-1}((z, \zeta)) = \{(z, \zeta, x, \eta)\}$  for  $(z, \zeta, x, \eta) \in \mathcal{C}$ , then  $D\pi_L|_{(z, \zeta, x, \eta)}$  is injective.

Then we have a decomposition of  $\mathcal{R}^*\mathcal{R}$  of the form

$$\mathcal{R}^*\mathcal{R} = A + \sum_{k=1}^{n''} \sum_{\alpha \in \Lambda_k} F_{k, \alpha},$$

where  $A$  is an elliptic PsDO of order  $-n'$  on  $X$ ,  
 $F_{k, \alpha}$  is a FIO in  $\mathcal{I}^{-(n+1-k)/2}(X \times X, \mathcal{C}'_{A_{k, \alpha}}; \Omega_{X \times X}^{1/2})$  with some canonical relation of  $\mathcal{C}_{F_{k, \alpha}}$ ,  
associated to the decomposition of connected components  $C_{R, k} = \bigcup_{\alpha \in \Lambda_k} C_{R, k, \alpha}$ .

# Outline of the proof

- $\mathcal{R}^*\mathcal{R}$  is given by

$$\frac{\mathcal{R}^*\mathcal{R}f}{|dX|^{1/2}}(x) = \iint_{H_x \times G_z} \overline{\kappa(z, x)} \kappa(z, y) \frac{f}{|dX|^{1/2}}(y) dG_z(y) dH_x(z).$$

- Follow the idea of Holman and Uhlmann [2]: a partition of unity of  $\mathcal{G} \times X \times X$ .
- Set  $C_\delta := \{(z; x, x) : z \in \mathcal{G}, x \in X\}$ , which is related to the elliptic term.
- $C_{R,k,\alpha}$  are disjoint since  $C_S = \emptyset$ , so are  $C_{R,k,\alpha}$  and  $C_\delta$ .  
Pick up disjoint nbds  $U_{k,\alpha}$  and  $U_\delta$  of  $C_{R,k,\alpha}$  and  $C_\delta$  respectively in  $\mathcal{G} \times X \times X$ .
- We can find an open set  $U_0$  in  $\mathcal{G} \times X \times X$  such that

$$U_0 \cup U_\delta \cup (\cup U_{k,\alpha}) = \mathcal{G} \times X \times X, \quad U_0 \cap \left( C_\delta \cup (\cup C_{R,k,\alpha}) \right) = \emptyset.$$

- Pick up a partition of unity subordinated to  $\{U_0, U_\delta, U_{k,\alpha}\}$ , and split the Schwartz kernel of  $\mathcal{R}^*\mathcal{R}$ .  $U_0$ -part of  $\mathcal{R}^*\mathcal{R}$  is a smoothing operator, and is absorbed in  $U_\delta$ -part  $A$ .

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